The Kähler-Ricci Flow on Surfaces of Positive Kodaira Dimension.

Outline:

1. Introduction:
   (i) Kodaira Dimension, and Classification of surfaces
   (ii) Elliptic Surfaces, and elliptic Fibre bundles
2. Statement of the Theorem.
3. Estimates and qualitative behavior of the Flow.

Next Time: use estimates and geometry to prove the main Theorem.

Recall: Let S be a cpct cplx surface, and define $P_m(S) := \dim H^0(S, K_S^{\otimes m})$ where K_S denotes the Canonical Bundle. Then

(i) if $P_m(S) = 0 \ \forall m$ we set $\text{Kod}(S) = -1$

(ii) if $\lim_{m\to\infty} \frac{P_m(S)}{m} < \infty$ then $\text{Kod}(S) = 1$

(iii) if $\lim_{m\to\infty} \frac{P_m(S)}{m^2} < \infty$ then $\text{Kod}(S) = 2$

(iv) if $P_m(S) \neq 0 \ \forall m$, but $P_m(S)$ is bounded then $\text{Kod}(S) = 0$.

In this talk we are interested in Surfaces with $\mathrm{Kod}(X) > 0$. When $\mathrm{Kod}(S) = 2$, X is said to be of general type. Tsuji showed that the KRF on Minimal mfds of general type converges to a KE metric off a divisor, which defines a current over X. We are thus interested in the case when $\mathrm{Kod}(X) = 1$.

Theorem: if $\mathrm{Kod}(X) = 1$, then X is an elliptic surface.

Def'n: an Elliptic surface X is a cpct complex mfd X together with a holomorphic map $f: X \to \Sigma$ to a curve Σ such that the generic fibre $f^{-1}(s)$ is a smooth, irreducible elliptic curve.

The Fibres can be singular, reducible or multiple.

Examples of Elliptic Surfaces:

(1) Let Σ_1 be any curve and Σ_2 be an elliptic curve. Then $\Sigma_1 \times \Sigma_2$ is an elliptic surface

(2) A pencil of Elliptic Curves over $\mathbb{P}^1$

Let $F(\lambda,\mu,x,y,z) = \lambda y^2 z + \mu x(x^2 + axz + z^2)$

and consider the set $F(\lambda,\mu,x,y,z) = 0$.

Since $F(t\lambda, t\mu, x,y,z) = t\,F(\lambda,\mu,x,y,z)$

The set $\{F=0\}$ defines an elliptic curve over $\mathbb{P}^1$.

That is, $\{F=0\}$ defines a section of the trivial bundle $\begin{array}{c}\mathbb{C}^3\\ \downarrow\\ \mathbb{P}^1\end{array}$. Now, $F(~~\mu~~\lambda,\mu, tx,ty,tz) = t^3 F(\mu,\lambda,x,y,z)$

And so we get a divisor in the Total space $\mathrm{Proj}\left(\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}\right)$

$$\begin{array}{c}\mathrm{Proj}\left(\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}\right)\\ \downarrow f\\ \mathbb{P}^1\end{array}$$

Note that for $[\lambda;\mu] \neq [0;1], [1:0]$, $f^{-1}([\lambda:\mu])$ is an elliptic curve. ~~when Δz $[0:1] = [\lambda:\mu]$, then~~

~~f^{-1} the~~ Thus, X is an elliptic surface with two Singular Fibres.

(3) When X has no singular Fibres, it is called an elliptic Fibre bundle.

X is said to be Minimal if $f^{-1}(s)$ does not contain a (-1) curve for any $s \in ~~\mathbb{f}~~ \Sigma$.

we define Σ_{reg} to be the set of points $s \in X$, such that $f^{-1}(s)$ is a smooth elliptic curve. The Theorem we aim to prove is:

<u>Theorem 1.1</u>:

Let $f: X \to \Sigma$ be a minimal elliptic surface with $Kod(X) = 1$, and singular fibres $X_{s_1} = m_1 F_1, \ldots X_{s_k} = m_k F_k$ (here $X_{s_i} = f^{-1}(s_i)$) with multiplicity $m_i \in \mathbb{N}$, $i = 1, \ldots, k$.

<u>Then</u>, for any initial Kähler metric, the KRF

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = -Ric(\omega_t) - \omega_t \\ \omega(0) = \omega_0 \end{cases} \qquad (KRF)$$

has a global solution $\omega(t)$ $\forall\, t \in [0, \infty)$ satisfying

1) $\omega(t)$ converges to $f^*\omega_\infty \in -2\pi c_1(X)$ as currents for a positive ω_∞ on Σ;

2) ω_∞ is smooth on Σ_{reg} and satisfies as currents on Σ

$$Ric(\omega_\infty) = -\omega_\infty + \omega_{WP} + \sum_{i=1}^{k} \frac{m_k - 1}{m_k} [s_i]$$

where ω_{wp} is the induced Weil-Petersson metric and $[s_i]$ is the current of integration associated to the

divisor S_i on Σ.

3) For any cpct subset $K \subseteq X_{reg} = f^{-1}(\Sigma_{reg})$, there is a constant C_K such that

$$\|w(t,\cdot) - f^* w_\infty\|_{L^\infty(K)} + e^t \sup_{s \in K} \|w(t,\cdot)|_{f^{-1}(s)}\|_{L^\infty} \leq C_K.$$

Moreover, the scalar curvature of $w(t,\cdot)$ is uniformly bounded on any cpct set of X_{reg}.

The KRF on Minimal Elliptic Surfaces w/ Kod $= 1$.

We want to solve (KRF). The First step is to choose a good reference metric. Note:

$$\begin{cases} \frac{d}{dt}[w] = -2\pi c_1(X) - [w] \\ [w](0) = [w_0]. \end{cases}$$

Thus $[w](t) = -2\pi c_1(X) + e^{-t}([w_0] + 2\pi c_1(X))$

Now, since $\mathrm{Kod}(X) = 1$, For large m, $H^0(X, K_X^{\otimes m})$ gives rise to a map $X \xrightarrow{\Phi} \mathbb{P}^N$ with $\dim_{\mathbb{C}}(\Phi(X)) = 1$. We thus obtain a positive $(1,1)$ form χ such that $f^*\chi$ represents $-2\pi c_1(X)$.

By abuse of notation, we will denote $f^*\chi$ by χ.
Define $\omega_t = \chi + e^{-t}(\omega_0 - \chi)$. Then the solution of (KRF) is given by $\omega(t) = \omega_t + \sqrt{-1}\partial\bar\partial\varphi$. Let Ω be a volume form such that $Ric(\Omega) = -\chi$. Then φ satisfies

$$\begin{cases} \dot\varphi = \log\left[\dfrac{e^t(\omega_t + \sqrt{-1}\partial\bar\partial\varphi)^2}{\Omega}\right] - \varphi \\ \varphi(0) = 0 \end{cases} \qquad (*)$$

<u>Note</u>: the e^t term arises from the $\partial\bar\partial$ Lemma.
We choose this normalization as $\int_X e^t \omega^2 = \text{const}$.

It was proved by Tian, Tsuji etc that $(*)$ has a solution
~~It is important for our applications to choose a $\partial\bar\partial$ which we~~
on $[0,\infty)$ when K_X is NEF.

It is of particular importance to understand how χ reflects the geometry of the surface X.

<u>Theorem</u>: Let $f: X \to \Sigma$ be a minimal elliptic surface with multiple fibers $X_{s_1} = m_1 F_1, \ldots, X_{s_k} = m_k F_k$. $m_i \geq 2$.
<u>Then</u>

~~$K_X = f^*(L) \otimes \mathcal{O}_X(\sum (m_i - 1) F_i) = f^*(L) \otimes \mathcal{O}_X(\sum \frac{(m_i-1)}{m_i} f^*[s_i])$~~

$$K_X = f^*D + \sum_i (m_i - 1) F_i$$

For D a divisor with $\deg D = 2g(\Sigma) - 2 + \chi(\mathcal{O}_X)$

In particular, $K_X = f^*\left(D + \sum_i \frac{m_i - 1}{m_i} s_i\right)$

Thus if m is sufficiently large, then $mK = f^*L$ for a line bundle $L \to \Sigma$ of degree ~~$m(2g-2+\chi(\mathcal{O}_X)+\sum \frac{m_i-1}{m_i})$~~

$$m\left(2g - 2 + \chi(\mathcal{O}_X) + \sum_i \frac{m_i - 1}{m_i}\right).$$

In particular, we observe that the linear system $H^0(X, mK_X)$ for m sufficiently large maps the fibres of $f: X \to \Sigma$ to points! (since any $\psi \in H^0(X, mK_X)$ is constant on $f^{-1}(s)$ as mK_X is pulled back from Σ). Thus, we can assume that $\chi = f^*\tilde{\chi}$ for some $\tilde{\chi}$ on Σ. This already hints at the geometry we should expect in the limit!!

Estimates

(i) Zeroth order and Volume Estimates

Lemma 4.1

Let φ be a solution of (*). There exists C depending only on ω_0 s.t. $\varphi \leq C$.

Pf: At a max of φ we have

$\chi \wedge \chi = 0$ since $\chi|_{f^{-1}(s)} = 0$.

$$\frac{\partial \varphi_{max}}{\partial t} \leq \log\left(\frac{e^t \omega_t^2}{\Omega}\right) - \varphi_{max} = \log\left[e^t\left\{\chi\wedge\chi + 2e^{-t}\chi\wedge(\omega_0-\chi) + e^{-2t}(\omega_0-\chi)^2\right\}/\Omega\right] - \varphi_{max}$$

$$\leq \log\left(\frac{2\chi\wedge(\omega_0-\chi) + e^{-t}(\omega_0-\chi)^2}{\Omega}\right) - \varphi_{max} \leq C - \varphi_{max}.$$

Now apply the max. principle ∎

Lemma 4.2: $\exists\, C = C(\omega_0)$ s.t. $\frac{\partial\varphi}{\partial t} \leq C$.

Pf using (*) we compute

$$\frac{\partial}{\partial t}(\dot\varphi) = \Delta_\omega \dot\varphi + 1 - e^{-t}\mathrm{tr}_\omega(\omega_0-\chi) - \dot\varphi$$

So

$$\frac{\partial}{\partial t}(e^t\dot\varphi) = \Delta_\omega e^t\dot\varphi + e^t - \mathrm{tr}_\omega(\omega_0-\chi)$$

$\omega_t = \chi + e^{-t}(\omega_0-\chi) = (1-e^{-t})\chi + e^{-t}\omega_0$

~~using (**) $\dot\varphi \leq$~~

Also $\Delta\varphi = \mathrm{tr}_\omega(\partial\bar\partial\varphi) = \mathrm{tr}_\omega(\omega_t + \partial\bar\partial\varphi - \omega_t) = 2 - \mathrm{tr}_\omega(\chi) - e^{-t}\mathrm{tr}_\omega(\omega_0-\chi)$

≥ 0.

So $\frac{\partial}{\partial t}(\dot{\varphi}) = \Delta_\omega \dot{\varphi} + tr_\omega(\chi) - 1 + \Delta\varphi - \dot{\varphi}$

Thus $\frac{\partial}{\partial t}(\dot{\varphi}+\varphi) = \Delta_\omega(\dot{\varphi}+\varphi) + tr_\omega(\chi) - 1.$

Hence

$$\frac{\partial}{\partial t}(e^t\dot{\varphi} - \dot{\varphi} - \varphi - e^t - t) = \Delta(e^t\dot{\varphi} - \dot{\varphi} - \varphi - e^t - t) - tr_\omega(\omega_0)$$

Now apply maximum principle to see

$$\dot{\varphi} \leq \frac{e^{-t}}{1-e^{-t}}\varphi + C' \leq C \quad \blacksquare$$

Lemma 4.3:

$\exists\, C = C(\omega_0)$ such that $|\varphi| \leq C$.

Pf set $u = \max_X \varphi(t,\cdot) - \varphi(t,z) \geq 0$

From the KRF eq'n, $\omega^2 \leq Ce^{-t}\omega_0^2$ by the upper bounds For $\varphi, \dot{\varphi}$. Thus:

$$\int_X e^{p\delta u}(\omega^2 - \omega_t^2) \leq \int_X e^{p\delta u}\omega^2 \leq Ce^{-t}\int_X e^{p\delta u}\omega_0^2$$

Now: $\omega^2 - \omega_t^2 = (\omega - \omega_t)\wedge(\omega + \omega_t) = \sqrt{-1}\,(\partial\bar{\partial}(-u))\wedge\omega + \omega_t$

So $$\int_X e^{p\delta u}(\omega^2 - \omega_t^2) \geq \sqrt{-1}\int_X e^{p\delta u}\left(\partial\bar{\partial}(-u)\wedge(\omega + \omega_t)\right)$$

$$= \frac{\sqrt{-1}\,2}{p\delta}\int_X \partial\left(e^{\frac{p\delta}{2}u}\right)\wedge\bar\partial\left(e^{\frac{p\delta}{2}u}\right)\wedge(\omega+\omega_t)$$ Now $\omega+\omega_t \geq \omega_t$

so

$$\geq \frac{\sqrt{-1}\,C e^{-t}}{p\delta}\int_X \left|\nabla e^{\frac{p\delta}{2}u}\right|^2 \omega_0$$

Thus

$$\int_X \left|\nabla e^{\frac{p\delta}{2}u}\right|^2 \omega_0 \leq C\int_X \left(e^{\frac{p\delta}{2}u}\right)^2 \omega_0$$

so we can apply the Moser Iteration. We obtain a C^0 bound for u provided $\exists\, \delta>0$ s.t.

$$\int e^{p\delta u}\,\omega_0 < C$$ For some $\delta>0$, C depending on (X,ω_0). we need:

Proposition 4.1:

$\exists\, \delta>0$ and C depending only on (X,ω_0) such that

$$\int_X e^{-\delta\varphi}\,\omega_0^n \leq C \quad \forall\, \varphi\in C^2(X) \text{ with } \omega_0+i\partial\bar\partial\varphi>0$$

and $\sup_X \varphi = 0$.

Proof ?

This Finishes the proof of Lemma 4.3.

<u>Note</u>: we have $e^{\varphi+\dot\varphi}\Omega = e^t\omega^2$, so $e^t\omega^2$ is Bounded above, and a Lower bound will Follow From a Lower bound for $\dot\varphi$.

<u>Lemma 4.4</u>:

$\exists\ \lambda_1, C > 0$ s.t. $\frac{1}{C}|S|_h^{2\lambda_1} \leq \frac{e^t\omega^2}{\Omega} \leq C$

For h a Fixed metric on $\mathcal{O}_X([S])$ such that $Ric(h)$ is a multiple of χ.

<u>Remark</u>: Such an h exists since, For m Large [S] $m[S]$ is pulled back From Σ. i.e. $\exists\ L \to \Sigma$ Line bundle s.t. $f^*L = m[S]$. Fix any metric h on L. Then $Ric_\Sigma(h) \in H^{1,1}(\Sigma, \mathbb{R})$ which is 1 dimensional. Also, $\chi \in H^{1,1}(\Sigma, \mathbb{R})$. Thus, $\exists\ \lambda \in \mathbb{R}$, $\varphi \in C^\infty(\Sigma, \mathbb{R})$ such that $Ric_\Sigma(h) = \lambda\chi + \partial\bar\partial\varphi$ <u>so</u> we can pull back the metric $he^{+\varphi}$ on L to a metric on $m[S]$. Then $h^{\frac{1}{m}}e^{\varphi/m}$ defines a metric on $[S]$ with Ricci curvature a multiple of $f^*\chi$.

<u>pf</u> of Lemma 4.4

we compute $\left(\frac{\partial}{\partial t} - \Delta\right)\log\frac{e^t\omega^2}{\Omega} = tr_\omega(\chi) - 1$

and $\left(\frac{\partial}{\partial t} - \Delta\right)\varphi = tr_\omega(\omega_t) + \log \frac{e^t \omega^2}{\Omega} - \varphi - 2$

So For any $\lambda > 0$ we have:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\log\left(\frac{e^t\omega^2}{\Omega}\right) + 2A\varphi - \lambda \log |S|_h^2\right)$$

$$\geq A\, tr_\omega(\omega_t) + 2A \log\left(\frac{e^t \omega^2}{\Omega}\right) - C(A+1)$$

By choosing A Large enough so that $tr_\omega(A\omega_t - \lambda Ric(h)) \geq 0$

Suppose that $\max_{X \times [0,T]} \log \frac{e^t \omega^2}{\Omega} + 2A\varphi - \lambda \log |S|_h^2$ is achieved at (t_0, z_0). Then we have (by the max. principle)

$$tr_\omega(\omega_t)(t_0, z_0) \lesssim C + 2\log\left(\frac{\Omega}{e^t\omega^2}\right)$$

But

$$tr_\omega(\omega_{t_0}) \geq \left(\frac{\omega_{t_0}^2}{\omega^2}\right)^{\frac{1}{2}} \geq \left(\frac{\Omega}{e^{t_0}\omega^2}\right)^{\frac{1}{2}}\left(\frac{e^{t_0}\omega_{t_0}^2}{\Omega}\right)^{\frac{1}{2}}$$

$$\geq \left(\frac{\Omega}{e^{t_0}\omega^2}\right)^{\frac{1}{2}}\left(\frac{2\chi \wedge \omega_0}{\Omega}\right)^{\frac{1}{2}} \geq C\left(|S|_h^{2\lambda}\frac{\Omega}{e^{t_0}\omega^2}\right)^{\frac{1}{2}}$$

↑ b/c $z_0 \in X \setminus S$. and we only need it at (t_0, z_0).

→ Note $z_0 \notin [S]$ since $\log |S|_h^2 \equiv -\infty$ on $[S]$ and so Max is in $X \setminus S$.

Now: $\forall \delta > 0\ \exists C_\delta$ s.t. $\log x < x^\delta + C_\delta$ for $x > 0$.

It follows that at (z_0, t_0):

$$\left(|S|^{2\lambda} \frac{\Omega}{e^{t_0}\omega^2} \right)^{1/2} \leq C \left(\left(\frac{\Omega}{e^{t_0}\omega^2} \right)^\delta + 1 \right) \quad \text{for some } \delta < \tfrac{1}{2}$$

multiplying by $|S|_h^{2\delta\lambda_1}$ we obtain

$$\left(|S|_h^{2\lambda + 4\delta\lambda_1} \frac{\Omega}{e^{t_0}\omega^2} \right)^{1/2} \leq C \left(\left(\frac{|S|_h^{2\lambda_1}\Omega}{e^{t_0}\omega^2} \right)^\delta + 1 \right)$$

choose $\lambda_1 = \dfrac{\lambda}{1-2\delta}$ so that $2\lambda + 4\delta\lambda_1 = 2\lambda_1$. Then

$$\left(|S|_h^{2\lambda_1} \frac{\Omega}{e^{t_0}\omega^2} \right)^{1/2} \leq C \left(\left(\frac{|S|_h^{2\lambda_1}\Omega}{e^{t_0}\omega^2} \right)^\delta + 1 \right)$$

Now $x^{1/2} \leq C(x^\delta + 1)$ for $\delta < \frac{1}{2} \Rightarrow x < K(\delta, C)$

Thus
$$\boxed{|S|_h^{2\lambda_1} \frac{\Omega}{e^{t_0}\omega^2} \leq C.}$$
Since (z_0, t_0) was chosen to be the minimum of $\log \frac{e^t\omega^2}{\Omega} + 2A\varphi - \lambda \log |S|_h^2$ we have

$$\frac{e^t\omega^2}{|S|_\Omega^{2\lambda_1}} e^{\varphi}(t,z) \geq \frac{e^{t_0}\omega^2}{|S|_h^{2\lambda_1}\Omega} e^{\varphi}(t_0, z_0) \geq C^{-1}$$

Now, C^{-1} is independent of time, and so the Lemma follows. ∎

Lemma 4.5:

$\exists\, C>0$ s.t. $\dot{\varphi} \geq C(\lambda, \log|S|_h^2 - 1)$

pf. $\dot{\varphi} = \log \frac{e^t \omega^2}{\Omega} - \varphi$. Now use Lemma 4.4.

Partial Second order Estimate and Qualitative Behavior

Lemma 4.6

For any $\delta>0$ $\exists\, C_\delta>0$ s.t. $tr_\omega(\chi) \leq \frac{C_\delta}{|S|_h^{2\delta}}$

pf

Set $u = tr_\omega \chi = g^{i\bar{j}} f_i^\alpha f_{\bar{j}}^{\bar{\beta}} \chi_{\alpha\bar{\beta}}$. Then one computes

$\left(\frac{\partial}{\partial t} - \Delta\right) \log |S|^{2\delta} u - A\varphi \leq -Au - 3A\dot{\varphi} + CA$. For A large

~~By Lemma~~ Thus, at the maximum of u:

$u \leq -3\dot{\varphi} + C \Rightarrow |S|^{2\delta} u \leq -3\dot{\varphi}|S|^{2\delta} + C$

$\leq C$ by Lemma 4.5 ($\Rightarrow \dot{\varphi}|S|^{2\delta} > -C$ for any $\delta>0$)

Corollary 4.1:

Let X_s be a

Corollary 4.1

Let X_s be a non singular Fibre for any $s \in \Sigma_{reg}$. Then along the KRF, ω decays exponentially fast on X_s. Furthermore, if Δ_s is the Laplacian on X_s wrt $\omega_0|_{X_s}$, then $\exists\ \lambda_2 > 0$ and $C > 0$ s.t.

$$-e^{-t} \leq \Delta_s \varphi \leq \frac{Ce^{-t}}{|s|^{2\lambda_2(s)}}$$

pf

$$0 < \mathrm{tr}_{\omega_0|_{X_s}}(\omega|_{X_s}) = \frac{\omega|_{X_s}}{\omega_0|_{X_s}} = \frac{\omega \wedge \chi}{\omega_0 \wedge \chi} = \frac{\omega \wedge \chi}{\omega^2} \frac{\omega^2}{\omega_0 \wedge \chi}$$

$$\leq (\mathrm{tr}_\omega \chi) \frac{\omega^2}{\omega_0 \wedge \chi} \leq \frac{Ce^{-t}}{|s|^{2\lambda_2(s)}} \quad \text{by Lemma 4.6}$$

This proves the Corollary. ∎

∴ we see that the KRF contracts the smooth Fibres of the ~~[illegible]~~ elliptic surface!!

Moreover, one can show:

Corollary 4.2' $\exists\, C, \lambda_3 > 0$ s.t. $\forall\, s \in \Sigma_{reg}$.

$$\left|\sup_{X_s} \varphi - \inf_{X_s} \varphi\right| \leq \frac{Ce^{-t}}{|S|_h^{2\lambda_3}(s)}$$

Gradient Estimates:

Let $u = \dot{\varphi} + \varphi = \log\left(\frac{e^t \omega^2}{\Omega}\right) - Th$

Theorem 4.1 $\exists\, \lambda_4, \lambda_5$ and $C > 0$ s.t.

(i) $|S|_h^{2\lambda_4} |\nabla u|^2 \leq C(A-u)$

(ii) $-|S|_h^{2\lambda_5} \Delta u \leq C(A-u)$.

Corollary 4.3: by Lemma 4.5, $\forall\, \delta > 0$ $|S|_h^{\delta} \dot{\varphi} > -C_\delta$.

Hence $\forall\, \delta > 0$ $\exists\, C_\delta$ s.t.

(i) $|S|_h^{2\lambda_4 + \delta} |\nabla u|^2 \leq C_\delta$

(ii) $-|S|_h^{2\lambda_5 + \delta} \Delta u \leq C_\delta$

Thus, we obtain:

Corollary 4.4:

The scalar curvature R is uniformly bounded on any cpct set of X_{reg}. More precisely, $\exists\ \lambda_6 > 0$ and C s.t.

$$-C \leq R \leq \frac{C}{|S|_h^{2\lambda_6}}$$

pf

The lower bound follows from the evolution eq'n for R and the max. principle. For the upper bound:

$$R_{i\bar{j}} = -u_{i\bar{j}} - \chi_{i\bar{j}} \Rightarrow R = -\Delta u - tr_\omega \chi$$

Thus the result follows from the partial 2nd order estimate in Lemma 4.6 and Corollary 4.3 ∎